Isaac Pliskin
Physics 859 | Homework 2 | Problem 1

1.) A particle of charge $q_e = e$ is confined to a circular ring of radius R.

A ~~magnetic~~ monopole of magnetic charge $q_m = N\tilde{e}$, $N \in \mathbb{Z}$ is located along the central axis of the ring, a distance z from the center.

(a.) Find the energy eigenspectrum of the charged particle ~~[illegible]~~ as a function of z.

For a monopole at the origin, we had two possible vector potentials:

$$\vec{A}_+ = \frac{q_m}{4\pi r}\left[\frac{1-\cos\theta}{\sin\theta}\right]\hat{\phi}$$

$$\vec{A}_- = \frac{-q_m}{4\pi r}\left[\frac{1+\cos\theta}{\sin\theta}\right]\hat{\phi}$$

where $\vec{A}_+$ is defined on $\mathcal{U}_+$ and $\vec{A}_-$ defined on $\mathcal{U}_-$, with $\mathcal{U}_\pm$ given by

$$\mathcal{U}_+ = \mathbb{R}^3 - (\{-z\} \cup \{0\})$$

$$\mathcal{U}_- = \mathbb{R}^3 - (\{+z\} \cup \{0\})$$

We need to choose $\mathcal{U}_\pm$ based on if the charged particle is above or below the monopole.

Actually, it should not matter, as the charge is restricted to a ring of radius ~~ϕ~~ R, so it will never reach the z axis.

~~Let us shift so that the ~~[illegible]~~ ring is sitting in the xy-plane at ~~[illegible]~~ $z=0$.)~~

~~To do this, let's go to cylindrical coordinates:~~

~~● $\rho = r\sin\theta$~~ | ~~● [illegible] =~~ ~~$x = \rho\cos\phi$~~

~~$\phi = \phi$~~ ~~[illegible]~~ ~~$y = \rho\sin\phi$~~

~~$z = r\cos\theta$~~ ~~[illegible]~~ ~~$z = z$~~

This gives

~~$r = \sqrt{\rho^2+z^2}$~~

~~$\cos\theta = z/r = z/\sqrt{\rho^2+z^2}$~~

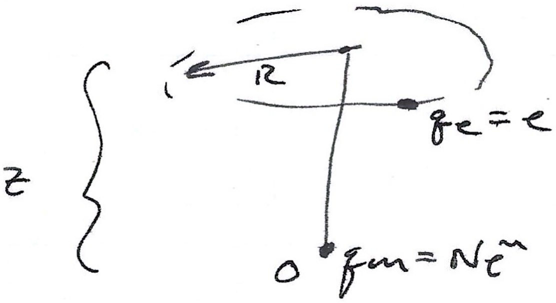
~~$\sin\theta = \rho/r = \rho\sqrt{\rho^2+z^2}$~~

~~ρ is fixed so that ~~ρ~~ $r=R$ implies~~

~~$\rho = R\sin\theta$~~

With

~~[illegible]~~ place the ring at a position z above the northpole.



~~[illegible]~~ Then, we can use $\mathcal{U}_+$ and $\vec{A}_+$:

$$\mathcal{U}_+ = \mathbb{R}^3 - (\{-z\} \cup \{0\})$$
$$\vec{A}_+ = \frac{q_m}{4\pi r}\left[\frac{1-\cos\theta}{\sin\theta}\right]\hat{\phi}$$

Switch to cylindrical coordinates

$$\rho = r\sin\theta$$
$$\phi = \phi$$
$$z = r\cos\theta$$

~~[illegible]~~

$\rho = R$ is fixed so we have

$$R = r\sin\theta \implies r = \frac{R}{\sin\theta}$$

Thus,

$$z = r\cos\theta = \left(\frac{R}{\sin\theta}\right)\cos\theta = \frac{R}{\tan\theta}$$

I don't think this is useful...

We also have

$$r = \sqrt{\rho^2 + z^2}$$

But $\rho = R$ gives

$$r = \sqrt{R^2 + z^2}$$

Thus,

$$\vec{A}_+ = \frac{g_m}{4\pi r}\left[\frac{1 - \cos\theta}{\sin\theta}\right]\hat{\phi}$$

$$= \frac{g_m}{4\pi}\frac{1}{r\sin\theta}\left[1 - \frac{z}{r}\right]\hat{\phi}$$

$$= \frac{g_m}{4\pi R}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{\phi}$$

$$\boxed{\vec{A}_+(z) = \frac{g_m}{4\pi R}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{\phi}}$$

Which is just a function of z, as needed.

In cylindrical coordinates, $\vec{\nabla}$ is given by

$$\vec{\nabla} = \hat{\rho}\,\partial_\rho + \frac{1}{\rho}\hat{\phi}\,\partial_\phi + \vec{z}\,\partial_z$$

$\vec{\nabla}$ will be independent of ρ and z so we just need the ϕ part:

$$\vec{\nabla} = \frac{1}{\rho}\hat{\phi}\,\partial_\phi$$

$$\boxed{\vec{\nabla} = \frac{1}{R}\hat{\phi}\,\partial_\phi}$$

since $\rho = R$ is fixed.

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The Hamiltonian is given by

$$H = \frac{1}{2m}(\vec{p} - q_e\vec{A})^2 + q_e\Phi$$

But $\Phi = 0$ so we are left with

$$H = \frac{1}{2m}(\vec{p} - q_e\vec{A})^2$$

To quantize we will replace $\vec{p} \to -i\vec{\nabla}$

We have to be a bit careful since $\vec{A}_+$ is a function of z.

Expand out $\hat{H}$:

$$\hat{H} = \frac{1}{2m}(-i\vec{\nabla} - q_e\vec{A}_+)^2$$

$$= \frac{1}{2m}(-i\vec{\nabla} - q_e\vec{A}_+)\cdot(-i\vec{\nabla} - q_e\vec{A}_+)$$

$$= \frac{1}{2m}(-\nabla^2 + iq_e\vec{\nabla}\cdot\vec{A}_+ + iq_e\vec{A}_+\cdot\vec{\nabla} + q_e^2\vec{A}_+^2)$$

$$\vec{\nabla}\cdot A_+ = \cancelto{0}{\partial_\rho A_{+\rho}} + \frac{1}{\rho}\partial_\phi A_{+\phi} + \cancelto{0}{\partial_z A_{+z}}$$

$$= \frac{1}{R}\partial_\phi\left[\frac{q_m}{4\pi R}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\right]$$

$$= 0.$$

So the worry was unfounded.

$$\vec{A}_+\cdot\vec{\nabla} = A_{+\phi}\left(\frac{1}{R}\partial_\phi\right)$$

$$= \frac{q_m}{4\pi R}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\frac{1}{R}\partial_\phi$$

$$= \frac{q_m}{4\pi R^2}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\partial_\phi$$

$$\vec{A}_+^2 = \frac{q_m}{4\pi R}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{\phi}\cdot\frac{q_m}{4\pi R}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{\phi}$$

$$= \left(\frac{q_m}{4\pi R}\right)^2\underbrace{\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]^2}_{\text{focus on this.}}$$

$$\left[1-\frac{z}{\sqrt{R^2+z^2}}\right]^2 = \left[1-\frac{z}{\sqrt{R^2+z^2}}\right]\left[1-\frac{z}{\sqrt{R^2+z^2}}\right]$$

$$= 1 - \frac{z}{\sqrt{R^2+z^2}} - \frac{z}{\sqrt{R^2+z^2}} + \frac{z^3}{R^2+z^2}$$

$$= 1 + \frac{z^2}{R^2+z^2} - 2\frac{z}{\sqrt{R^2+z^2}}$$

$$= \frac{R^2\cancel{+z^2}}{R^2+z^2} + \frac{\cancel{z^2}}{R^2+z^2} - 2\frac{z}{\sqrt{R^2+z^2}}$$

$$= \frac{R^2+2z^2}{R^2+z^2} - \frac{2z\sqrt{R^2+z^2}}{R^2+z^2}$$

$$= \frac{1}{R^2+z^2}\left[R^2+2z^2-2z\sqrt{R^2+z^2}\right]$$

$$= \frac{1}{R^2+z^2}\left[(R^2+z^2)+z^2-2z\sqrt{R^2+z^2}\right]$$

$$= \frac{1}{R^2+z^2}\left[\left(z-\sqrt{R^2+z^2}\right)\left(z-\sqrt{R^2+z^2}\right)\right]$$

$$= \frac{1}{R^2+z^2}\left(z-\sqrt{R^2+z^2}\right)^2.$$

$$= \frac{1}{r^2}(z-r)^2$$

where for ease we go back to $r=\sqrt{R^2+z^2}$ for now.

Thus,

$$\hat{H} = \frac{1}{2m}\left[-\nabla^2 + iq_e\frac{q_m}{4\pi R^2}\left[1-\frac{z}{r}\right]\partial_\phi + q_e^2\left(\frac{q_m}{4\pi R^2}\right)^2\frac{1}{r^2}(z-r)^2\right]$$

$$= \frac{1}{2m}\left[-\nabla^2 + \frac{iq_eq_m}{4\pi R^2}\left[1-\frac{z}{r}\right]\partial_\phi + \left(\frac{q_eq_m(z-r)}{4\pi rR^2}\right)^2\right]$$

To find the eigenspectrum we need to solve

$$\hat{H}\Psi = E\Psi$$

$$\Rightarrow \frac{1}{2m}\left[-\nabla^2 + \frac{iq_eq_m}{4\pi R^2}\left[1-\frac{z}{r}\right]\partial_\phi + \left(\frac{q_eq_m(z-r)}{4\pi rR^2}\right)^2\right]\Psi = E\Psi$$

$$\left[-\nabla^2 + \frac{iq_eq_m}{4\pi R^2}\left[1-\frac{z}{r}\right]\partial_\phi + \left(\frac{q_eq_m(z-r)}{4\pi rR^2}\right)^2\right]\Psi = 2mE\Psi$$

$$-\nabla^2 \Psi + \frac{i q_e q_m}{4\pi R^2}\left[1-\frac{z}{r}\right]\partial_\phi \Psi = 2mE\,\Psi - \left(\frac{q_e q_m (z-r)}{4\pi r R^2}\right)^2 \Psi$$

$$= \left[2mE - \left(\frac{q_e q_m (z-r)}{4\pi r R^2}\right)^2\right]\Psi$$

Thus,

$$-\frac{1}{R^2}\partial_\phi^2 \Psi + \frac{i q_e q_m}{4\pi R^2}\left[1-\frac{z}{r}\right]\partial_\phi \Psi = \left[2mE - \left(\frac{q_e q_m}{4\pi r R^2}(z-r)\right)^2\right]\Psi$$

Note that $q_e = e$, $q_m = N\tilde{e}$ and

$$e\tilde{e} = 2\pi$$

So that

$$q_e q_m / 4\pi = Ne\tilde{e}/4\pi$$
$$= 2\pi N / 4\pi$$
$$= N/2 .$$

Giving

$$-\frac{1}{R^2}\partial_\phi^2 \Psi + \frac{iN}{2R^2}\left[1-\frac{z}{r}\right]\partial_\phi \Psi = \left[2mE - \left(\frac{N}{2}\cdot\frac{(z-r)}{rR^2}\right)^2\right]\Psi$$

$$\partial_\phi^2 \Psi - \frac{iN}{2}\left[1-\frac{z}{r}\right]\partial_\phi \Psi = \left[R^2\left(\frac{N}{2}\frac{(z-r)}{rR^2}\right)^2 - 2mER^2\right]\Psi$$

$$= \left[\left(\frac{N}{2}\frac{(z-r)}{rR}\right)^2 - 2mER^2\right]\Psi$$

$$= \tilde{E}\,\Psi$$

with $\tilde{E} = \left[\left(\frac{N}{2}\frac{(z-r)}{rR}\right)^2 - 2mER^2\right]$

Also let $f(z) = \frac{N}{2}\left[\frac{z}{r}-1\right]$ so that we

$$\boxed{\partial_\phi^2 \Psi + i f(z)\, \partial_\phi \Psi = \tilde{E}\,\Psi}$$

Let us assume that $\Psi(\phi) \propto e^{\lambda\phi}$ for some λ a constant.

Then,

$$\partial_\phi^2 \Psi = \partial_\phi^2\left[e^{\lambda\phi}\right]$$
$$= \partial_\phi\left[\lambda e^{\lambda\phi}\right]$$

$$= \lambda^2 e^{\lambda\phi}$$

And $\partial_\phi \Psi = \lambda e^{\lambda\phi}$

Thus, we find

$$\lambda^2 \Psi + i f(z) \lambda \Psi = \tilde{E} \Psi$$

$$\lambda^2 \Psi + i f(z) \lambda \Psi - \tilde{E} \Psi = 0$$

$$(\lambda^2 + i f(z) \lambda - \tilde{E}) \Psi = 0$$

Since $\Psi \neq 0$, we have

$$\lambda^2 + i f(z) \lambda - \tilde{E} = 0$$

$$\Rightarrow \lambda_{\pm} = \frac{-(i f(z)) \pm \sqrt{(i f(z))^2 - 4(1)(-\tilde{E})}}{2(1)}$$

$$\cancel{-(i f(z))^2 = -(i^2) f^2 = -(-1) f^2 = f^2}$$

$$(if)^2 - 4(1)(-\tilde{E}) = -f^2 + 4\tilde{E}$$

$$= 4\tilde{E} - f^2$$

$$= 4\left(\left[\frac{N}{2}\frac{(z-r)}{rR}\right]^2 - 2mER^2\right) - \left[\frac{N}{2}\left(\frac{z}{r}-1\right)\right]^2$$

Note that

$$\frac{N}{2}\frac{(z-r)}{rR} = \frac{N}{2}\left(\frac{z-r}{r}\right)\frac{1}{R}$$

$$= \frac{N}{2}\left(\frac{z}{r} - \frac{r}{r}\right)\frac{1}{R}$$

$$= \frac{N}{2}\left[\frac{z}{r} - 1\right]\frac{1}{R}$$

$$= f(z)\frac{1}{R}$$

So we have

$$\tilde{E} = \left[\left(\frac{N}{2}\frac{(z-r)}{rR}\right)^2 - 2mER^2\right]$$

$$= \left[(f/R)^2 - 2mER^2\right]$$

$$\Rightarrow (if)^2 - (4)(1)(-\tilde{E}) = 4\tilde{E} - f^2$$

$$= 4\left[\left(\frac{f}{R}\right)^2 - 2mER^2\right] - f^2$$

Ah, this feels wrong because I messed up the $f(z)$!

~~I forgot the $\frac{1}{R}$ in the ∂_ϕ term.~~

I have an extra factor of $1/R$ in the ~~[illegible]~~ $\vec{A}_+^2$ term.

$$\vec{A}_+^2 = \left(\frac{g_m}{4\pi R}\right)^2\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]^2$$

$$= \left(\frac{g_m}{4\pi R}\right)^2\left(\frac{z-r}{r}\right)^2$$

$$= \left(\frac{g_m}{4\pi R}\right)^2\left(\frac{z}{r} - 1\right)^2$$

So that we actually have

$$-\frac{1}{\cancel{R^2}}\partial_\phi^2\Psi + \frac{iN}{2\cancel{R^2}}\left[1 - \frac{z}{r}\right]\partial_\phi\Psi = \left[2mE - \left(\frac{N}{2}\frac{(z-r)}{r\cancel{R}}\right)^2\right]\Psi$$

$$\partial_\phi^2\Psi + \frac{iN}{2}\left[\frac{z}{r} - 1\right]\partial_\phi\Psi = \left[-2mER^2 + \left(\frac{N}{2}\left(\frac{z}{r} - 1\right)\right)^2\right]\Psi$$

$$\partial_\phi^2\Psi + if(z)\partial_\phi\Psi = \left[f(z)^2 - 2mER^2\right]\Psi$$

$$\Rightarrow \lambda_\pm = \frac{-if \pm \sqrt{-f^2 + 4(f^2 - 2mER^2)}}{2}$$

$$= \frac{1}{2}\left[-if \pm \sqrt{4f^2 - 8mER^2 - f^2}\right]$$

$$= \frac{1}{2}\left[-if \pm \sqrt{3f^2 - 8mER^2}\right]$$

With $f(z) = \frac{N}{2}\left[\frac{z}{r} - 1\right]$

Thus,

$$\Psi(\phi) = c_+ e^{\lambda_+\phi} + c_- e^{\lambda_-\phi}$$

with $\lambda_\pm = \frac{1}{2}\left[-if \pm \sqrt{3f^2 - 8mER^2}\right]$

$$= -\frac{i}{2}f \pm \frac{1}{2}\sqrt{3f^2 - 8mER^2}$$

$$\Rightarrow \Psi(\phi) = c_+ e^{-\frac{i}{2}f\phi} e^{+\frac{1}{2}\sqrt{3f^2 - 8mER^2}\,\phi} + c_- e^{-\frac{i}{2}f\phi} e^{-\frac{1}{2}\sqrt{3f^2 - 8mER^2}\,\phi}$$

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$$\Psi(\phi) = e^{-\frac{i}{2}f\phi}\left[C_+ e^{\frac{1}{2}\sqrt{3f^2-8mER^2}\phi} + C_- e^{-\frac{1}{2}\sqrt{3f^2-8mER^2}\phi}\right]$$

We need $\Psi(\phi) = \Psi(\phi+2\pi)$

$\Psi'(\phi) = \Psi'(\phi+2\pi)$

~~Let me redefine $\lambda = \frac{1}{2}\sqrt{3f^2-8mER^2}$ so that~~

~~$\lambda_\pm = -\frac{if}{2} \pm \lambda$~~

~~Then,~~

~~$\Psi(\phi) = e^{-\frac{if}{2}\phi}\left[C_+ e^{+\lambda\phi} + C_- e^{-\lambda\phi}\right]$~~

~~$\Psi(\phi+2\pi) = e^{-\frac{if}{2}(\phi+2\pi)}\left[C_+ e^{\lambda(\phi+2\pi)} + C_- e^{-\lambda(\phi+2\pi)}\right]$~~

~~$= e^{-\frac{if}{2}\phi}\left[C_+ e^{\lambda\phi} e^{\lambda 2\pi} + C_- e^{-\lambda\phi} e^{-\lambda 2\pi}\right]$~~

~~$= e^{-\frac{if}{2}\phi}\left[C_+ e^{\lambda\phi} + C_- e^{-\lambda\phi}\right]$~~

~~$\Rightarrow C_+ e^{2\pi\lambda} + C_- e^{-2\pi\lambda} = C_+ + C_-$~~

~~Similarly, we find~~

~~$\Psi'(\phi+2\pi) =$~~

More simply, we need to impose that

$$\Psi(0) = \Psi(2\pi)$$
$$\Psi'(0) = \Psi'(2\pi)$$

Plugging in to $\Psi(\phi)$ we have

$$\Psi(0) = e^{-\frac{i}{2}f(0)}\left[C_+ e^{\lambda(0)} + C_- e^{-\lambda(0)}\right]$$

where I redefined

$$\lambda_\pm = -\frac{if}{2} \pm \lambda \longrightarrow \lambda = \frac{1}{2}\sqrt{3f^2+8mER^2}$$

Thus,

$$\Psi(0) = e^{-0}[c_+ e^{0} + c_- e^{0}]$$

$$= c_+ + c_-$$

$$\Psi(2\pi) = e^{-\frac{i}{2}f(2\pi)}[c_+ e^{2\pi\lambda} + c_- e^{-2\pi\lambda}]$$

$$= e^{-i\pi f}[c_+ e^{2\pi\lambda} + c_- e^{-2\pi\lambda}]$$

$f(z) = \frac{N}{2}[\frac{z}{r} - 1]$ is not an integer in general, so we cannot get rid of the $e^{-i\pi f}$ term.

However,

$$e^{-i\pi f} = \exp\{-i\pi \tfrac{N}{2}[\tfrac{z}{r} - 1]\}$$

$$= \exp\{-i\tfrac{\pi}{2}\tfrac{Nz}{r}\}\exp\{\tfrac{i\pi N}{2}\}$$

And $e^{i\pi/2} = i \Rightarrow \exp\{\frac{i\pi N}{2}\} = i^N$.

Although this seems to complicate things.

We have

$$c_+ + c_- = e^{-i\pi f}[c_+ e^{2\pi\lambda} + c_- e^{-2\pi\lambda}]$$

The derivative gives

$$\Psi'(\phi) = \frac{d}{d\phi}[e^{-\frac{i}{2}f\phi}[c_+ e^{\lambda\phi} + c_- e^{-\lambda\phi}]\}$$

A simpler solution can be found from the following.

We want to solve

$$\hat{H}\Psi = E\Psi.$$

with

$$\hat{H} = \frac{1}{2m}(-i\vec{\nabla} - q_e\vec{A})^2 + q_e\Phi$$

Since $\Phi = 0$, the Hamiltonian reduces to

$$\hat{H} = \frac{1}{2m}(-i\vec{\nabla} - q_e\vec{A})^2$$

So we see that $\hat{H}$ is equal to the square of another operator

$$\hat{H} = \frac{1}{2m}(-i\vec{\nabla} - q_e\vec{A})^2$$

$$= \frac{1}{2m}(-i\vec{\nabla} + (-1)q_e\vec{A})^2$$

$$= \frac{1}{2m}(-i\vec{\nabla} + (i)iq_e\vec{A})^2$$

$$= \frac{1}{2m}[(-i)[\vec{\nabla} - iq_e\vec{A}]]^2$$

$$= \frac{1}{2m}[-i\vec{D}]^2$$

So $\hat{H}$ shares eigenstates with the operator $-i\vec{D}$.

Thus, we need to just solve

$$-i\vec{D}\Psi = \vec{k}\Psi$$

with $\vec{D} = \vec{\nabla} - iq_e\vec{A}$.

For our problem we have $q_e = e$, and Ψ is independent of ρ, z in cylindrical coordinates:

$$\partial_\rho\Psi = 0$$
$$\partial_z\Psi = 0.$$

Thus,

$$\vec{D} = \vec{\nabla} - ie\vec{A}$$

And our equation becomes

$$-i(-ieA_\rho\Psi) = k_\rho\Psi$$
$$-i(-ieA_z\Psi) = k_z\Psi$$
$$-i\left(\left(\underset{1/R}{\partial_\phi} - ieA_\phi\right)\right)\Psi = k_\phi\Psi$$

Our vector potential is

$$\vec{A} = \vec{A}_+(z)$$

$$= \frac{q_m}{4\pi R}\left[1 - \frac{z}{r}\right]\hat{\phi}$$

with $r = \sqrt{R^2+z^2}$ as before.

Again, $q_m = N\tilde{e}$, so $\vec{A}_+$ is given by

$$\vec{A}_+ = \frac{N\tilde{e}}{4\pi R}\left[1 - \frac{z}{r}\right]\hat{\phi}$$

Note that this gives

$$A_{+\rho} = A_{+z} = 0$$

implying that

$$k_\rho \Psi = k_z \Psi = 0$$

$$\Rightarrow \boxed{k_\rho = k_z = 0}$$

So we are left with one equation

$$\left(\frac{1}{R}\partial_\phi - ieA_\phi\right)\Psi = ik_\phi \Psi$$

Redefine $k_\phi \rightarrow k$ since there is only one k.

Thus, we arrive at

$$\left[\frac{1}{R}\partial_\phi - ie\frac{N\tilde{e}}{4\pi R}\left(1 - \frac{z}{r}\right)\right]\Psi = ik\Psi$$

$$\frac{1}{R}\left[\partial_\phi - i\frac{Ne\tilde{e}}{4\pi}\left(1 - \frac{z}{r}\right)\right]\Psi = ik\Psi$$

From class we know that

$$\tilde{e} = \frac{2\pi}{e}$$

$$\Rightarrow \tilde{e}e = 2\pi$$

So our equation is

$$\left(\partial_\phi - i\frac{N2\pi}{4\pi}\left(1 - \frac{z}{r}\right)\right)\Psi = iRk\Psi$$

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$$\left(\partial_\phi - \frac{iN}{2}\left(1-\frac{z}{r}\right)\right)\Psi = iRk\Psi$$

or, by moving over the second term,

$$\partial_\phi \Psi = iRk\Psi + \frac{iN}{2}\left(1-\frac{z}{r}\right)\Psi$$
$$= \left(iRk + \frac{iN}{2}\left(1-\frac{z}{r}\right)\right)\Psi$$
$$= i\lambda\Psi$$

with $\boxed{\lambda = Rk + \frac{N}{2}\left(1-\frac{z}{r}\right)}$

We can solve this simply with

$$\partial_\phi \Psi = i\lambda\Psi$$
$$\Rightarrow \Psi = Ce^{i\lambda\phi}$$

For some constant C.

k is fixed through the condition that Ψ is continuous

$$\Psi(0) = \Psi(2\pi)$$
$$Ce^{i\lambda(0)} = Ce^{i\lambda 2\pi}$$
$$C = Ce^{i\lambda 2\pi}$$
$$C(1-e^{i\lambda 2\pi}) = 0$$
$$\Rightarrow 1-e^{i\lambda 2\pi} = 0$$
$$1 = e^{i\lambda 2\pi}$$
$$\cancel{\lambda 2\pi = \log(1) = 0}$$

I missed the $-i$

$$\cancel{\Rightarrow \lambda = 0}$$
$$\cancel{\Rightarrow Rk + \frac{iN}{2}\left(1-\frac{z}{r}\right) = 0}$$
$$\cancel{Rk = \frac{iN}{2}\left(\frac{z}{r}-1\right)}$$
$$\cancel{\boxed{k = \frac{iN}{2R}\left(\frac{z}{r}-1\right)}}$$

$$1 = e^{2\pi i \lambda}$$

$$\Rightarrow \lambda \in \mathbb{Z}$$

$$\lambda = n$$

$$= Rk + \frac{N}{2}\left(1 - \frac{z}{r}\right)$$

$$Rk + \frac{N}{2}\left(1 - \frac{z}{r}\right) = n$$

$$Rk = n + \frac{N}{2}\left(\frac{z}{r} - 1\right)$$

$$k = \frac{N}{2R}\left(\frac{z}{r} - 1 + \frac{2n}{N}\right)$$

~~Redefine~~ $n \longrightarrow Nn$

$$\xcancel{\boxed{k = \frac{N}{2R}\left(\frac{z}{r} - 1 + n\right)}}$$

$$\boxed{k_n = \frac{N}{2R}\left[\frac{z}{r} - 1 + \frac{2n}{N}\right]}$$

for $n \in \mathbb{Z}$.

C is fixed through normalization.

$$R\int_0^{2\pi} d\phi \, |\Psi|^2 = 1$$

But $\Psi(\phi) = Ce^{i\lambda\phi} \Rightarrow |\Psi|^2 = |C|^2$

$$\Rightarrow R\int_0^{2\pi} d\phi |C|^2 = 1$$

$$= |C|^2 \int_0^{2\pi} d\phi$$

$$= 2\pi |C|^2 R$$

So we can take

$$C = \frac{1}{\sqrt{2\pi R}}$$

Thus we find that

$$\boxed{\Psi_n(\phi) = \frac{1}{\sqrt{2\pi R}} \exp\{in\phi\}}$$

which satisfies

$$\boxed{-iD_\phi\Psi_n = \frac{N}{2R}\left[\frac{z}{r} - 1 + \frac{2n}{N}\right]\Psi_n}$$

The Hamiltonian thus gives

$$\begin{aligned}\hat{H}\Psi_n = E_n\Psi_n &= \frac{1}{2m}(i\vec{D})^2\Psi_n \\ &= \frac{1}{2m}(-iD_\phi)^2\Psi_n \\ &= \frac{1}{2m}\left[\frac{N}{2R}\left(\frac{z}{r} - 1 + \frac{2n}{N}\right)\right]^2\Psi_n\end{aligned}$$

So that

$$E_n = \frac{1}{2m}\left[\frac{N}{2R}\left(\frac{z}{r} - 1 + \frac{n}{N}\right)\right]^2$$

$$\boxed{E_n = \frac{N^2}{8R^2m}\left[\frac{z}{r} - 1 + \frac{2n}{N}\right]^2}$$

So our Eigenspectrum is given by

$$\boxed{\begin{aligned}\Psi_n(\phi) &= \frac{1}{\sqrt{2\pi R}}e^{in\phi} \\ E_n &= \frac{N^2}{8R^2m}\left[\frac{z}{\sqrt{R^2+z^2}} - 1 + 2\frac{n}{N}\right]^2\end{aligned}}$$

□

(b.) The monopole is slowly moved along the central axis from $z=-\infty$ to $z=+\infty$, passing through the center of the ring.

If the charged particle is initially in its ground state, what state does it end up in, assuming the adiabatic approximation?

[This is a nice example of spectral flow.]

Adiabatic Approximation: assume the system tracks a single energy level as the monopole moves, rather than jumping between levels.

From part a, we found that the states are

$$\psi_n = \frac{1}{\sqrt{2\pi R}} e^{in\phi}$$

$$E_n = \frac{N^2}{8mR^2}\left[\frac{z}{\sqrt{R^2+z^2}} - 1 + 2\frac{n}{N}\right]^2$$

The particle starts in the ground state, i.e. $n=0$:

$$\psi_0 = \frac{1}{\sqrt{2\pi R}} e^{i(0)\phi} = \frac{1}{\sqrt{2\pi R}} e^0$$

$$= \frac{1}{\sqrt{2\pi R}} (1)$$

$$= \frac{1}{\sqrt{2\pi R}}$$

$$E_0 = \frac{N^2}{8mR^2}\left[\frac{z}{\sqrt{R^2+z^2}} - 1 + \overset{0}{\cancel{2\frac{(0)}{N}}}\right]^2$$

$$= \frac{N^2}{8mR^2}\left[\frac{z}{\sqrt{R^2+z^2}} - 1\right]^2$$

This allows us to rewrite E_n as

$$E_n = \frac{N^2}{8mR^2}\left[\frac{z}{\sqrt{R^2+z^2}} - 1 + 2\frac{n}{N}\right]^2$$

$$= \frac{N^2}{8mR^2}\left\{\left[\frac{z}{\sqrt{R^2+z^2}} - 1\right]^2 + \left[2\frac{n}{N}\right]^2 + 2\left(2\frac{n}{N}\right)\left(\frac{z}{\sqrt{R^2+z^2}} - 1\right)\right\}$$

$$= \underbrace{\frac{N^2}{8mR^2}\left[\frac{z}{\sqrt{R^2+z^2}} - 1\right]^2}_{E_0} + \frac{N^2}{8mR^2}\left(2\frac{n}{N}\right)\left(2\frac{n}{N} + 2\left[\frac{z}{\sqrt{R^2+z^2}} - 1\right]\right)$$

$$= E_0 + \frac{\cancel{N^2}1}{\cancel{2}\cancel{8}mR^2}\cancel{2}\frac{n}{\cancel{N}}\left(\cancel{4}\frac{\cancel{2}}{\cancel{N}}\right)\left[n + N\left(\frac{z}{\sqrt{R^2+z^2}} - 1\right)\right]$$

$$= E_0 + \frac{n}{2mR^2}\left[n + N\left(\frac{z}{\sqrt{R^2+z^2}} - 1\right)\right]$$

$$\boxed{E_n = E_0 + \frac{n}{2mR^2}\left[n + N\left(\frac{z}{\sqrt{R^2+z^2}} - 1\right)\right]}$$

As the monopole moves through the ring there will be a flux through the area of the ring.

First, we need to shift to a coordinate system where the ring is at $z=0$ instead of the monopole.

This amounts to shifting $z \longrightarrow -z$ so that

$$\psi_n(\phi) = \frac{1}{\sqrt{2\pi R}} e^{in\phi}$$

$$E_n(z) = E_0 + \frac{n}{2mR^2}\left[n \cancel{+} - N\left(\frac{z}{\sqrt{R^2+z^2}} + 1\right)\right]$$

$$E_0 = \frac{N^2}{8mR^2}\left[\frac{z}{\sqrt{z^2+R^2}} + 1\right]^2$$

$$\vec{A}_+ = \frac{g_m}{4\pi R}\left[1 + \frac{z}{\sqrt{z^2+R^2}}\right]\hat{\phi}$$

$$g_m = N\tilde{e}$$

The flux through the ring as a function of z is given by

$$\Phi_B(z) = \oint \vec{A}\cdot d\vec{r}$$

$$= \oint \vec{A}_+ \cdot d\vec{r}$$

$$= \int_0^{2\pi} A_{+\phi} R\, d\phi$$

$$= R\int_0^{2\pi}\left[\frac{g_m}{4\pi R}\left(1 + \frac{z}{\sqrt{z^2+R^2}}\right)\right] d\phi$$

$$= \cancel{R}\,\frac{N\tilde{e}}{4\pi \cancel{R}}\left[1 + \frac{z}{\sqrt{z^2+R^2}}\right] \underbrace{\int_0^{2\pi} d\phi}_{=2\pi}$$

$$= \frac{N\tilde{e}}{2\,\cancel{4\pi}}(\cancel{2\pi})\left[1 + \frac{z}{\sqrt{z^2+R^2}}\right]$$

$$= \frac{N\tilde{e}}{2}\left[1 + \frac{z}{\sqrt{z^2+R^2}}\right]$$

$$= \frac{N}{\cancel{2}}\left(\frac{\cancel{2}\pi}{e}\right)\left[1 + \frac{z}{\sqrt{z^2+R^2}}\right].$$

So we can see that in our expression for E_n,

$$E_{n} = E_0 + \frac{n}{2mR^2}\left[n - N\left(\frac{z}{\sqrt{R^2+z^2}} + 1\right)\right]$$

~~$= E_0 + \frac{n}{mR^2}\frac{e}{2\pi}\left(\frac{1}{2}\right)n$~~

~~$= E_0 + \frac{n}{mR^2}\frac{e}{2\pi}\left(\frac{2\pi}{2e}\right)\left[n - N\left(\frac{z}{\sqrt{R^2+z^2}} + 1\right)\right]$~~

$$= E_0 + \frac{n}{2mR^2}\left[n - \frac{e}{\pi}\left(\frac{N\pi}{e}\right)\left(\frac{z}{\sqrt{z^2+R^2}} + 1\right)\right]$$

$$= E_0 + \frac{n}{2mR^2}\left[n - \frac{e}{\pi}\Phi_B(z)\right].$$

Furthermore,

$$\vec{A}_+ = \frac{\cancel{2\pi}N}{2\,\cancel{4\pi}eR}\left[1 + \frac{z}{\sqrt{z^2+R^2}}\right]\hat{\phi}$$

$$= \frac{N}{2eR}\left[1 + \frac{z}{\sqrt{z^2+R^2}}\right]\hat{\phi}$$

$$= \frac{1}{2\pi R}\left[\left(\frac{N\pi}{e}\right)\left(1 + \frac{z}{\sqrt{z^2+R^2}}\right)\right]\hat{\phi}$$

$$= \frac{1}{2\pi R}\Phi_B(z)\hat{\phi}$$

$$\boxed{\vec{A}_+ = \frac{1}{2\pi R}\Phi_B(z)\hat{\phi}}$$

$$\boxed{E_n = E_0 + \frac{n}{2mR^2}\left[n - \frac{e}{\pi}\Phi_B(z)\right]}$$

And also

$$E_0 = \frac{N^2}{8mR^2}\left[\frac{z}{\sqrt{z^2+R^2}}+1\right]^2$$

$$= \frac{1}{8mR^2}\frac{e^2}{\pi^2}\left[\left(\frac{N\pi}{e}\right)\left(\frac{z}{\sqrt{z^2+R^2}}+1\right)\right]^2$$

$$= \frac{1}{8mR^2}\frac{e^2}{\pi^2}\Phi_B(z)^2$$

$$\boxed{E_0(z) = \frac{e^2}{8\pi^2 mR^2}\left[\Phi_B(z)\right]^2}$$

Something feels off about the second term of E_n ---

Originally I had

$$E_n = \frac{N^2}{8R^2m}\left[\frac{z}{\sqrt{R^2+z^2}} - 1 + 2\frac{n}{N}\right]^2$$

Shift $z \to -z$:

$$E_n = \frac{N^2}{8R^2m}\left[\frac{-z}{\sqrt{R^2+z^2}} - 1 + 2\frac{n}{N}\right]^2$$

$$= \frac{N^2}{8R^2m}\left[1 + \frac{z}{\sqrt{R^2+z^2}} - 2\frac{n}{N}\right]^2$$

We found that

$$\Phi_B(z) = \frac{N\pi}{e}\left[1 + \frac{z}{\sqrt{R^2+z^2}}\right]$$

$$\Rightarrow 1 + \frac{z}{\sqrt{z^2+R^2}} = \frac{e\Phi_B}{N\pi}$$

Which allows us to write

$$E_n = \frac{N^2}{8R^2m}\left[\frac{e\Phi_B}{N\pi} - 2\frac{n}{N}\right]^2$$

$$= \frac{N^2}{8R^2m}\left[\left(\frac{2}{N}\right)\left(\frac{e}{\pi}\Phi_B - n\right)\right]^2$$

$$= \frac{N^2}{8R^2m}\left(\frac{2}{N}\right)^2\left[\frac{e}{\pi}\Phi_B - n\right]^2$$

$$= \frac{\cancel{N^2}}{2\,\cancel{8}R^2m}\left(\frac{4}{\cancel{N^2}}\right)\left[\frac{e}{\pi}\Phi_B - n\right]^2$$

$$= \frac{1}{2R^2 m}\left[\frac{e}{\pi}\Phi_B - n\right]^2$$

$$= \frac{1}{2R^2 m}\left(\left(\frac{e}{\pi}\Phi_B\right)^2 + n^2 - 2\frac{e}{\pi}n\Phi_B\right)$$

Now,

$$E_0 = \frac{e^2}{8\pi^2 m R^2}\left[\Phi_B\right]^2$$

$$= \frac{1}{8mR^2}\left(\frac{e}{\pi}\Phi_B\right)^2$$

$$\Rightarrow \left(\frac{e}{\pi}\Phi_B\right)^2 = \cancel{\frac{1}{8mR^2}E_0}\ 8mR^2E_0$$

so that

$$E_n = \frac{1}{2R^2 m}\left[\left(\frac{e}{\pi}\Phi_B\right)^2 + n^2 - 2\frac{e}{\pi}n\Phi_B\right]$$

$$= \frac{1}{2R^2 m}\left[8mR^2E_0 + n^2 - 2\frac{e}{\pi}n\Phi_B\right]$$

$$= 4\frac{8mR^2}{\cancel{2}R^2 m}E_0 + \frac{1}{2R^2 m}\left[n^2 - 2\frac{e}{\pi}n\Phi_B\right]$$

$$= 4E_0 + \frac{1}{2R^2 m}\left[n^2 - 2\frac{e}{\pi}n\Phi_B\right]$$

Something is definitely wrong here. We have ~~that~~

~~$\Psi_n(\phi) = \frac{1}{\sqrt{2\pi R}}e^{in\phi}$~~

~~And~~

~~$H\Psi_n = E_n\Psi_n$~~

~~with~~

~~$H = \frac{1}{2m}(-i\vec{D})^2$~~

~~$\vec{D} = D_\phi\hat{\phi}$~~

~~$D_\phi = \frac{1}{R}\partial_\phi - \frac{iq_eq_m}{4\pi R}\left(1 + \frac{z}{\sqrt{R^2+z^2}}\right)$~~

$q_e = e$, $q_m = N\hat{e} = N\frac{2\pi}{e} \Rightarrow q_eq_m = e\left(N\frac{2\pi}{e}\right)$

$= 2\pi N$

~~Thus,~~

~~$$D_\phi = \frac{1}{R}\partial_\phi - \frac{2\pi i N}{2\,4\pi R}\left(1 + \frac{z}{\sqrt{R^2+z^2}}\right)$$~~

~~$$= \frac{1}{R}\partial_\phi - \frac{iN}{2R}\left(1 + \frac{z}{\sqrt{R^2+z^2}}\right)$$~~

~~$$= \frac{1}{R}\left[\partial_\phi - i\frac{N}{2}\left(1 + \frac{z}{\sqrt{R^2+z^2}}\right)\right]$$~~

~~Thus,~~

~~$$\mathcal{H}\Psi_n = \left\{\frac{1}{R}\left[\partial_\phi - i\frac{N}{2}\left(1 + \frac{z}{\sqrt{R^2+z^2}}\right)\right]\right\}^2 \Psi_n$$~~

~~$$= E_n \Psi_n$$~~

~~$$\Psi_n = \frac{1}{\sqrt{2\pi R}} e^{in\phi} \Rightarrow \partial_\phi \Psi_n = in\Psi_n$$~~

~~So we have~~

~~$$\frac{1}{R}\left[in - i\frac{N}{2}\left(1 + \frac{z}{\sqrt{R^2+z^2}}\right)\right]\Psi_n = E_n\Psi_n$$~~

~~$$\Rightarrow E_n =$$~~

~~But Ψ_n satisfies~~

~~$$-iD_\phi \Psi_k = k\Psi_k(\phi)$$~~

I think I messed up part (a.)...

We had

$$\mathcal{H}\Psi_k = E_k\Psi_k$$

But $\mathcal{H} = \frac{1}{2m}(-i\vec{D})^2$, so we can see that this is solved by Ψ_k such that

$$-i\vec{D}\Psi_k = \vec{k}\Psi_k$$

Ψ_k is only dependent on ϕ so this reduces to

$$-iD_\phi\Psi_k(\phi) = k_\phi\Psi_k = k\Psi_k$$

Where

$$D_\phi = \frac{1}{R}\partial_\phi - \frac{iN}{2R}\left[1 + \frac{z}{\sqrt{z^2+R^2}}\right]$$

Thus,

$$-iD_\phi \Psi_k = -i\left[\frac{1}{R}\partial_\phi - \frac{iN}{2R}\left(1+\frac{z}{\sqrt{R^2+z^2}}\right)\right]\Psi_k$$

$$= \frac{-i}{R}\Psi_k' - \frac{N}{2R}\left(1+\frac{z}{\sqrt{R^2+z^2}}\right)\Psi_k$$

$$= k\Psi_k$$

$$\Rightarrow -i\Psi_k' - \frac{N}{2}\left(1+\frac{z}{r}\right)\Psi_k = kR\Psi_k$$

$$\Psi_k' - \frac{iN}{2}\left(1+\frac{z}{r}\right)\Psi_k = +ikR\Psi_k$$

$$\Psi_k' = \left\{\frac{iN}{2}\left(1+\frac{z}{r}\right)+ikR\right\}\Psi_k$$

$$= i\left\{\frac{N}{2}\left(1+\frac{z}{r}\right)+kR\right\}\Psi_k$$

$$= i\lambda\Psi_k$$

$$\Rightarrow \boxed{\Psi_k'(\phi) = \mathcal{N}_k e^{i\lambda_k\phi}}$$

$$\boxed{\lambda_k = \frac{N}{2}\left(1+\frac{z}{r}\right)+kR}$$

So far this is what I had before.

Normalization gives

$$\int_0^{2\pi} R\,d\phi\,|\Psi|^2 = 1$$

$$\Rightarrow \mathcal{N} = \frac{1}{\sqrt{2\pi R}}$$

Periodicity gives

$$\Psi_k(0) = \Psi_k(2\pi)$$

$$\Rightarrow 1 = e^{2\pi i\lambda_k}$$

Thus,

$$\lambda_k = n \in \mathbb{Z}$$

Implying that

$$n = \frac{N}{2}\left(1 + \frac{z}{r}\right) + kR$$

$$\frac{n}{R} = \frac{N}{2R}\left(1+\frac{z}{r}\right) + k$$

$$\boxed{k_n = \frac{1}{R}\left(n - \frac{N}{2}\left(1+\frac{z}{r}\right)\right)}$$

which is useful since

$$-iD_\phi \Psi_n = k_n \Psi_n$$

Helps us solve

$$\mathcal{H}\Psi_n = E_n \Psi_n$$

$$\mathcal{H} = \frac{1}{2m}(-iD_\phi)^2 \Psi_n$$

$$= \frac{1}{2m}(k_n)^2 \Psi_n$$

$$\Rightarrow E_n = \frac{k_n^2}{2m}$$

But we need to add the zero point energy for the particle in the ring:

$$\boxed{E_n = \frac{k_n^2}{2m} + E_0}$$

Thus,

$$\boxed{\begin{aligned} \Psi_n(\phi) &= \frac{1}{\sqrt{2\pi R}} e^{in\phi} \\ E_n &= \frac{1}{2mR^2}\left(n - \frac{N}{2}\left[1 + \frac{z}{\sqrt{R^2+z^2}}\right]\right)^2 + E_0 \end{aligned}}$$

So what I was missing was this zero point energy for the particle on the ring.

Now I can continue ——

The magnetic flux through the ring is given by

$$\Phi_B = \oint d\vec{a}\cdot\vec{A}$$

with $\vec{A} = \vec{A}_+ = \frac{g_m}{4\pi R}\left[1 + \frac{z}{\sqrt{z^2+R^2}}\right]\hat{\phi}$

Thus,

$$\Phi_B = \int_0^{2\pi} \cancel{R}\,d\phi \left\{\frac{g_m}{4\pi \cancel{R}}\left(1 + \frac{z}{r}\right)\right\}$$

$$= \frac{g_m}{4\pi}\left(1+\frac{z}{r}\right)\int_0^{2\pi} d\phi$$

$$= \frac{g_m}{2\,4\pi}\left(1+\frac{z}{r}\right)\cancel{(2\pi)}$$

$$= \frac{N\cancel{2}\pi}{\cancel{2}e}\left(1+\frac{z}{r}\right)$$

$$\boxed{\Phi_B = \frac{N\pi}{e}\left(1+\frac{z}{r}\right)}$$

$$\Rightarrow \vec{A}_+ = \frac{g_m}{4\pi R}\left(1+\frac{z}{r}\right)\hat{\phi}$$

$$= \frac{2\pi N}{2\,4\pi e R}\left(1+\frac{z}{r}\right)\hat{\phi}$$

$$= \frac{N}{2eR}\left(1+\frac{z}{r}\right)\hat{\phi}$$

$$= \frac{1}{2\pi R}\left[\frac{N\pi}{e}\left(1+\frac{z}{r}\right)\right]\hat{\phi}$$

$$= \frac{1}{2\pi R}\Phi_B(z)\hat{\phi}$$

$$\boxed{\vec{A}_+ = \frac{1}{2\pi R}\Phi_B(z)\hat{\phi}}$$

Similarly,

$$E_n = E_0 + \frac{1}{2mR^2}\left[n - \frac{N}{2}\left(1+\frac{z}{r}\right)\right]^2$$

$$= E_0 + \frac{1}{2mR^2}\left[n - \frac{1}{2}\frac{e}{\pi}\left(\frac{N\pi}{e}\left\{1+\frac{z}{r}\right\}\right)\right]^2$$

$$= E_0 + \frac{1}{2mR^2}\left[n - \frac{e}{2\pi}\Phi_B\right]^2$$

So that

$$\boxed{E_n = E_0 + \frac{1}{2mR^2}\left[n - \frac{e}{2\pi}\Phi_B(z)\right]^2}$$

We see that the only part of the energy which changes with z is $\Phi_B(z)$, so let's examine that.

As $z \to \pm\infty$ we find

$$\lim_{z\to\infty}\Phi_B(z) = \lim_{z\to\infty}\left[\frac{N\pi}{e}\left(1 + \frac{z}{\sqrt{R^2+z^2}}\right)\right]$$

$$= \frac{N\pi}{e} + \frac{N\pi}{e}\lim_{z\to\infty}\left\{\frac{z}{\sqrt{R^2+z^2}}\right\}$$

$$= \frac{N\pi}{e} + \frac{N\pi}{e}\lim_{z\to\infty}\left(\frac{z}{\sqrt{z^2}}\right)$$

$$= \frac{N\pi}{e} + \frac{N\pi}{e}(1)$$

$$= \frac{N\pi}{e} + \frac{N\pi}{e}$$

$$= 2\frac{N\pi}{e}$$

$$\lim_{z\to-\infty}\Phi_B(z) = \lim_{z\to-\infty}\left\{\frac{N\pi}{e}\left(1 + \frac{z}{\sqrt{z^2+R^2}}\right)\right\}$$

$$= - - -$$

$$= \cancel{\frac{N\pi}{e}} + \cancel{\frac{N\pi}{e}(-1)}$$

$$= 0$$

So we see that

$$\boxed{\begin{aligned}\lim_{z\to\infty}\Phi_B(z) &= \frac{2\pi N}{e} = g_m\\ \lim_{z\to-\infty}\Phi_B(z) &= 0\end{aligned}}$$

This leads to the following for the energy:

$$E_n(z) = E_0 + \frac{1}{2mR^2}\left[n - \frac{e}{2\pi}\Phi_B(z)\right]^2$$

So for $z \to \infty$:

$$\lim_{z\to\infty} E_n(z) = \lim_{z\to\infty}\left\{E_0 + \frac{1}{2mR^2}\left(n - \frac{e}{2\pi}\Phi_B(z)\right)^2\right\}$$

$$= E_0 + \frac{1}{2mR^2}\left(n - \frac{\cancel{e}}{\cancel{2\pi}}\frac{\cancel{2\pi} N}{\cancel{e}}\right)^2$$

$$= E_0 + \frac{1}{2mR^2}(n-N)^2$$

for $z \to -\infty$:

$$\lim_{z\to-\infty} E_n(z) = \lim_{z\to-\infty}\left\{E_0 + \frac{1}{2mR^2}\left(n - \frac{e}{2\pi}\Phi_B(z)\right)^2\right\}$$

$$= ---$$

$$= E_0 + \frac{1}{2mR^2}\left(n - \cancelto{0}{\frac{e}{2\pi}(0)}\right)^2$$

$$= E_0 + \frac{1}{2mR^2}(n)^2$$

$$= E_0 + \frac{n^2}{2mR^2}$$

So we have

$$E_n(z \to -\infty) = E_0 + \frac{n^2}{2mR^2}$$

$$E_n(z \to +\infty) = E_0 + \frac{(n-N)^2}{2mR^2}.$$

which tells us that

$$\boxed{\Psi_n(\phi) \longrightarrow \Psi_{n-N}(\phi).}$$

As $z = -\infty \to z = +\infty$.

The ground state at $z = -\infty$ is $n = 0$, i.e.

$$E_{n=0}(z) = E_0 + \cancelto{0}{\frac{(0)^2}{2mR^2}}$$

$$= E_0$$

$\Rightarrow$ particle starts at E_0.

Thus,

$$\Psi_0(\phi) \longrightarrow \Psi_{0-N}(\phi) = \Psi_{-N}(\phi)$$

So that

$$\boxed{\Psi_0(\phi) \to \Psi_{-N}(\phi) = \frac{1}{\sqrt{2\pi R}} e^{-iN\phi}}$$

Next, note that

$$E_0 = E_N(z \to \infty)$$

So that the new ground state at $z \to \infty$ is indexed by $N = q_m/\tilde{e}$.

Thus, the new ground state at $z \to \infty$ is

$$\boxed{\Psi_N(\phi) = \frac{1}{\sqrt{2\pi R}} e^{iN\phi}}$$

This is indeed an example of spectral flow as ~~discusse~~ disscussed in class. □

(c.) Where does the energy gained (or lost) by the charged particle come from (or go)?

We saw that the energy changed by

$$E_0 \longrightarrow E_0 + \frac{N^2}{2mR^2}$$

So the change in energy is

$$\Delta E = \frac{N^2}{2mR^2} > 0$$

So we see the charged particle gains energy.

The monopole is moving, and the only thing that changes is the magnetic flux through the ring.

$$\Phi_B(z) = \frac{N\pi}{e}\left(1+\frac{z}{\sqrt{R^2+z^2}}\right)$$

We saw that

$$\Phi_B(z \to -\infty) = 0$$

$$\Phi_B(z \to +\infty) = \frac{2N\pi}{e}$$

We can recall Faraday's law:

$$\oint \vec{E}\cdot d\vec{r} = -\frac{d}{dt}\iint \vec{B}\cdot d\vec{a}$$

with $d\vec{a} \parallel +\hat{z}$ we see that this tells us that there is an EMF induced in the loop.

This EMF induces a magnetic field to cancel the increase in flux from the monopole. Call this $\vec{B}_{ind}$.

The force on the monopole is thus

$$\vec{F}_m = q_m \vec{B}_{ind} .$$

Now,

$$\oint \vec{E}\cdot d\vec{r} = \int_0^{2\pi} R\,d\phi\, E_\phi$$

$$= -\frac{d\Phi_B}{dz}\frac{dz}{dt}$$

So we need to find $\frac{d}{dz}\Phi_B$.

Note also that the particle has $dz/dt > 0$.

$$\Phi_B(z) = \frac{N\pi}{e}\left(1+\frac{z}{\sqrt{R^2+z^2}}\right)$$

$$\Rightarrow \frac{d\Phi_B}{dz} = \frac{d}{dz}\left\{\frac{N\pi}{e}\left(\cancelto{0}{1} + \frac{z}{\sqrt{R^2+z^2}}\right)\right\}$$

$$= \frac{N\pi}{e}\frac{d}{dz}\left\{\frac{z}{\sqrt{R^2+z^2}}\right\}$$

$$= \frac{N\pi}{e}\left\{\frac{d}{dz}(z)\frac{1}{\sqrt{R^2+z^2}} + z\frac{d}{dz}\left(\frac{1}{\sqrt{R^2+z^2}}\right)\right\}$$

$$= \frac{N\pi}{e}\left\{ \frac{1}{r} + z \frac{d}{dz}(R^2+z^2)^{-1/2} \right\}$$

$$= \frac{N\pi}{e}\left\{ \frac{1}{r} + z \left(-\frac{1}{2}\right)(R^2+z^2)^{-3/2} \frac{d}{dz}(\overset{\nearrow 0}{\cancel{R^2}}+z^2) \right\}$$

$$= \frac{N\pi}{e}\left\{ \frac{1}{r} + z \left(-\frac{1}{\cancel{2}}\right) r^{-3} (\cancel{2}z) \right\}$$

$$= \frac{N\pi}{e}\left\{ \frac{1}{r} - \frac{z^2}{r^3} \right\}$$

$$= \frac{N\pi}{e}\left\{ \frac{r^2-z^2}{r^3} \right\}$$

$$= \frac{N\pi}{e}\left\{ \frac{R^2}{r^3} \right\}$$

Now,

$$\oint \vec{E}\cdot d\vec{r} = -\frac{d\Phi_B}{dz}\frac{dz}{dt}$$

$$= -\underbrace{\frac{N\pi}{e}\left\{ \frac{R^2}{r^3} \right\}}_{>0} \underbrace{\frac{dz}{dt}}_{>0}$$

$$\Rightarrow \boxed{\oint \vec{E}\cdot d\vec{v} < 0}$$

$$\int_0^{2\pi} E_\phi R\, d\phi < 0$$

$$\Rightarrow E_\phi < 0 \quad \forall \phi$$

$$\Rightarrow \boxed{\vec{E} = -|E_\phi|\hat{\phi}}$$

Thus, $\vec{E}$ is in the $-\hat{\phi}$ direction!
From the Biot-Savart law, we
[illegible] the same

$$\vec{B}_{ind} = -|B_{z\,ind}|\hat{z}$$

$$\Rightarrow \boxed{\vec{F}_m = -q_m |B_{z\,ind}|\hat{z}}$$

We finally have that the work done is

$$W_m = \int \vec{F}_m \cdot d\vec{r}$$

$$= \underbrace{-q_m}_{>0} \underbrace{\int_{-\infty}^{\infty} |B_{z\,ind}|\, dz}_{>0}$$

$$\Rightarrow \boxed{W_m < 0}$$

Thus, the induced magnetic field causes the monopole to <u>lose</u> kinetic energy.

Thus, the energy <u>gained</u> by the charged particle comes from the kinetic energy <u>lost</u> by the monopole as it moves from $z=-\infty$ to $z=+\infty$.

□

3.) Show that

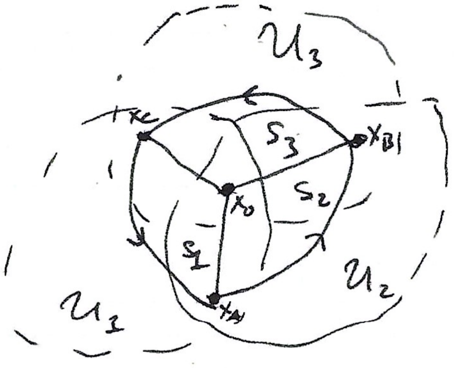
$$\exp\left\{i\frac{e}{\hbar}\int_S F\right\} = W_e(\partial S)$$

holds for an arbitrary gauge bundle, where $W_e(\partial\Sigma)$ is defined on a non-trivial gauge bundle in the manner discussed in lecture.

To do so, split S into pieces, each residing in a single patch and use Stokes' theorem.

Prove the formula in the following setup first, then explain in words how your proof generalizes to other situations:

Let us formalize the diagram above



Let $x_0 \in \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$.

$x_A \in \mathcal{U}_1 \cap \mathcal{U}_2$

$x_B \in \mathcal{U}_2 \cap \mathcal{U}_3$

$x_C \in \mathcal{U}_3 \cap \mathcal{U}_1$

$S = S_1 + S_2 + S_3$, $S_1 \subseteq \mathcal{U}_1$, $S_2 \subseteq \mathcal{U}_2$, $S_3 \subseteq \mathcal{U}_3$

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~~Let $\partial S = \Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$~~

Let $\partial S = \Sigma = \Sigma_{AB} + \Sigma_{BC} + \Sigma_{CA}$

where Σ_{ij} goes from x_i to x_j.

We can also define Σ_{OA}, Σ_{OB}, Σ_{OC} similarly.

Then, it is clear that

$$\partial S_1 = \Sigma_{CA} + \Sigma_{AO} + \Sigma_{OC}$$
$$= \Sigma_{CA} - \Sigma_{OA} + \Sigma_{OC}$$
$$\partial S_2 = \Sigma_{AB} + \Sigma_{BO} + \Sigma_{OA}$$
$$= \Sigma_{AB} - \Sigma_{OB} + \Sigma_{OA}$$
$$\partial S_3 = \Sigma_{BC} + \Sigma_{CO} + \Sigma_{OB}$$
$$= \Sigma_{BC} - \Sigma_{OC} + \Sigma_{OB}$$

(Where we have used the fact that

$$\Sigma_{Oi} = -\Sigma_{iO} \quad \text{for } i = A, B, C.$$

We can then see that

$$\partial S_1 + \partial S_2 + \partial S_3 = [\Sigma_{CA} - \Sigma_{OA} + \Sigma_{OC}] + [\Sigma_{AB} - \Sigma_{OB} + \Sigma_{OA}] + [\Sigma_{BC} - \Sigma_{OC} + \Sigma_{OB}]$$

$$= \Sigma_{CA} + \Sigma_{AB} + \Sigma_{BC} + \cancel{\Sigma_{OA}} - \cancel{\Sigma_{OA}} + \cancel{\Sigma_{OC}} - \cancel{\Sigma_{OC}} + \cancel{\Sigma_{OB}} - \cancel{\Sigma_{OB}}$$

$$= \Sigma_{CA} + \Sigma_{AB} + \Sigma_{BC}$$
$$= \Sigma_{AB} + \Sigma_{BC} + \Sigma_{CA}$$
$$= \partial S.$$

Thus,

$$\partial S_1 + \partial S_2 + \partial S_3 = \partial S$$

As needed.

Now,

$$\int_S F = \int_{S_1+S_2+S_3} F$$

$$= \int_{S_1} F + \int_{S_2} F + \int_{S_3} F$$

$$= \int_{S_1} dA_1 + \int_{S_2} dA_2 + \int_{S_3} dA_3 .$$

Thus, (setting $\hbar = 1$)

$$\exp\{ie\int_S F\} = \exp\{ie(\int_{S_1} dA_1 + \int_{S_2} dA_2 + \int_{S_3} dA_3)\}$$

$$= \exp\{ie\int_{S_1} dA_1 + ie\int_{S_2} dA_2 + ie\int_{S_3} dA_3\}$$

$$= \exp\{ie\int_{S_1} dA_1\}\exp\{ie\int_{S_2} dA_2\}\exp\{ie\int_{S_3} dA_3\}$$

Now, we apply Stokes' Theorem:

$$\int_{S_1} dA_1 = \oint_{\partial S_1} A_1$$

$$= \oint_{\Sigma_{CA} - \Sigma_{OA} + \Sigma_{OC}} A_1$$

$$= \int_{\Sigma_{CA}} A_1 + \int_{-\Sigma_{OA}} A_1 + \int_{\Sigma_{OC}} A_1$$

$$= \int_{\Sigma_{CA}} A_1 - \int_{\Sigma_{OA}} A_1 + \int_{\Sigma_{OC}} A_1$$

Similarly,

$$\int_{S_2} dA_2 = \int_{\Sigma_{AB}} A_2 - \int_{\Sigma_{OB}} A_2 + \int_{\Sigma_{OA}} A_2$$

$$\int_{S_3} dA_3 = \int_{\Sigma_{BC}} A_3 - \int_{\Sigma_{OC}} A_3 + \int_{\Sigma_{OB}} A_3$$

Thus,

$$\exp\left\{ie\int_S F\right\} = \exp\left\{ie\left(\int_{\Sigma_{CA}} A_1 - \int_{\Sigma_{OA}} A_1 + \int_{\Sigma_{OC}} A_1\right)\right\}$$

$$\times \exp\left\{ie\left(\int_{\Sigma_{AB}} A_2 - \int_{\Sigma_{OB}} A_2 + \int_{\Sigma_{OA}} A_2\right)\right\}$$

$$\times \exp\left\{ie\left(\int_{\Sigma_{BC}} A_3 - \int_{\Sigma_{OC}} A_3 + \int_{\Sigma_{OB}} A_3\right)\right\}$$

For ease, let $\int_{\Sigma_{ij}} := \int_{ij}$ so that

$$\exp\left\{ie\int_S F\right\} = \exp\left\{ie\left(\int_{CA} A_1 - \int_{OA} A_1 + \int_{OC} A_1\right)\right\}$$

$$\times \exp\left\{ie\left(\int_{AB} A_2 - \int_{OB} A_2 + \int_{OA} A_2\right)\right\}$$

$$\times \exp\left\{ie\left(\int_{BC} A_3 - \int_{OC} A_3 + \int_{OB} A_3\right)\right\}$$

$$= \exp\left\{ie\int_{CA} A_1 - ie\int_{OA} A_1 + \int_{OC} A_1\right\}$$

$$\times \exp\left\{ie\int_{AB} A_2 - ie\int_{OB} A_2 + \int_{OA} A_2\right\}$$

$$\times \exp\left\{ie\int_{BC} A_3 - ie\int_{OC} A_3 + ie\int_{OB} A_3\right\}$$

$$= \exp\left\{ie\int_{CA} A_1\right\}\exp\left\{-ie\int_{OA} A_1\right\}\exp\left\{ie\int_{OC} A_1\right\}$$

$$\times \exp\left\{ie\int_{AB} A_2\right\}\exp\left\{-ie\int_{OB} A_2\right\}\exp\left\{ie\int_{OA} A_2\right\}$$

$$\times \exp\left\{ie\int_{BC} A_3\right\}\exp\left\{-ie\int_{OC} A_3\right\}\exp\left\{ie\int_{OB} A_3\right\}$$

Since these are just numbers after integrating, we can rearrange as so:

$$= \exp\left\{ie\int_{CA} A_1\right\}\exp\left\{ie\int_{AB} A_2\right\}\exp\left\{ie\int_{BC} A_3\right\}$$

$$\times \exp\left\{ie\int_{OA} A_2\right\}\exp\left\{-ie\int_{OA} A_1\right\}$$

$$\times \exp\left\{ie\int_{OB} A_3\right\}\exp\left\{-ie\int_{OB} A_2\right\}$$

$$\times \exp\left\{ie\int_{OC} A_1\right\}\exp\left\{-ie\int_{OC} A_3\right\}$$

$$= \exp\left\{ie\int_{CA} A_1\right\}\exp\left\{ie\int_{AB} A_2\right\}\exp\left\{ie\int_{BC} A_3\right\}$$

$$\times \exp\left\{ie\int_{OA} A_2 - ie\int_{OA} A_1\right\}\exp\left\{ie\int_{OB} A_3 - ie\int_{OB} A_2\right\}\exp\left\{ie\int_{OC} A_1 - ie\int_{OC} A_3\right\}$$

~~$= \exp\{ie\int_{CA} A_1 + ie\int_{AB} A_2 + ie\int_{BC} A_3\}$~~

Let us investigate the last few terms.

$$\exp\{ie\int_{OA} A_2 - ie\int_{OA} A_1\} = \exp\{ie(\int_{OA} A_2 - \int_{OA} A_1)\}$$

$$= \exp\{ie\int_{OA}(A_2 - A_1)\}$$

But, we can relate A_2 and ~~A~~ A_1 using a gauge transformation since

$$\Sigma_{OA} \subseteq U_2 \cap U_1 .$$

Thus,

$$\cancel{A} \quad A_i = A_j + \frac{i}{e} g_{ij}\, dg_{ij}^{\dagger}$$

$$= A_j + d\lambda_{ij}$$

With $g_{ij} = \exp(ie\lambda_{ij})$ since $g_{ij} \in U(1)$.

Thus,

$$A_2 = A_1 + d\lambda_{21}$$

$$\Rightarrow A_2 - A_1 = \cancel{A_1} + d\lambda_{21} \cancel{- A_1}$$

$$= d\lambda_{21}$$

Thus,

$$\int_{OA} A_2 - A_1 = \int_{OA} d\lambda_{21}$$

$$= \lambda_{21}(x_0) - \lambda_{21}(x_A) .$$

$$\Rightarrow \exp\{ie\int_{OA}(A_2 - A_1)\} = \exp\{ie(\lambda_{21}(x_0) - \lambda_{21}(x_A))\}$$

$$= \exp\{ie\lambda_{21}(x_0) - ie\lambda_{21}(x_A)\}$$

$$= \exp\{ie\lambda_{21}(x_0)\}\exp\{-ie\lambda_{21}(x_A)\}$$

$$= g_{21}(x_0)\, g_{21}(x_A)^{-1}$$

$$= g_{21}(x_0)\, g_{21}(x_A)^\dagger$$

Thus,

$$\exp\{ie\textstyle\int_{0A} A_2 - ie\int_{0A} A_1\} = g_{21}(x_0)\, g_{21}(x_A)^\dagger$$

$$\exp\{ie\textstyle\int_{0B} A_3 - ie\int_{0B} A_2\} = g_{32}(x_0)\, g_{32}(x_B)^\dagger$$

$$\exp\{ie\textstyle\int_{0C} A_1 - ie\int_{0C} A_3\} = g_{13}(x_0)\, g_{13}(x_C)^\dagger .$$

Thus,

$$\exp\{ie\textstyle\int_S F\} = \exp\{ie\int_{CA} A_1\}\exp\{ie\int_{AB} A_2\}\exp\{ie\int_{BC} A_3\}$$

$$\times\, g_{21}(x_0)\, g_{21}(x_A)^\dagger g_{32}(x_0) g_{32}(x_B)^\dagger g_{13}(x_0) g_{13}(x_C)^\dagger$$

$$= g_{21}(x_0) g_{32}(x_0) g_{13}(x_0)\Big[\overset{\to g_{13}(x_C)}{\exp}\{ie\textstyle\int_{CA} A_1\}\, g_{12}(x_A)\exp\{ie\int_{AB} A_2\}$$

$$\times\, g_{23}(x_B)\exp\{ie\textstyle\int_{BC} A_3\}\Big]$$

$$= g_{21}(x_0) g_{32}(x_0) g_{13}(x_0)\, W_e(\partial S)$$

Where we used the definition of $W_e(\partial S)$ from class.

$$\Rightarrow \exp\{ie\textstyle\int_S F\} = g_{21}(x_0) g_{32}(x_0) g_{13}(x_0)\, W_e(\partial S)$$

But by definition,

$$g_{ij}\, g_{jk}\, g_{ke} = 1$$

on $U_i \cap U_j \cap U_k$.

Thus,

$$g_{21}(x_0) g_{32}(x_0) g_{13}(x_0) = 1$$

So indeed,

$$\boxed{\exp\{ie\textstyle\int_S F\} = W_e(\partial S)}$$

It is clear that if S lies in N patches, we can split S up as above, that is,

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$$S = \sum_i^N S_i$$

with $S_i \subseteq \mathcal{U}_i$ for each i, and

$$\partial S = \sum_i^N \partial S_i$$

with ∂S_i given by

$$\partial S_i = \Sigma_{ij} - \Sigma_{0i} + \Sigma_{0j}$$

with Σ_{ij} going from x_i to x_j, and $x_i \in \mathcal{U}_i \cap \mathcal{U}_{i-1}$, and x_0 in the intersection of all al $\mathcal{U}_i$'s.

Thus, we can write

$$\int_S F = \int_{\sum_i S_i} F = \sum_i \int_{S_i} F$$

$$= \sum_i \oint_{\partial S_i} A_i$$

And we will again get terms like

$$\int_{\Sigma_{i0}} (A_j - A_i) = \int_{\Sigma_{i0}} d\lambda_{ij} = \lambda_{ij}(x_j) - \lambda_{ij}(x_0)$$

Thus, we will get something like,

$$\exp\{ie\int_S F\} = \underbrace{\left(\prod_{j=1}^N g_{j,j-1}(x_0)\right)}_{=1} W_e(\partial S)$$

$$= W_e(\partial S). \quad \square$$

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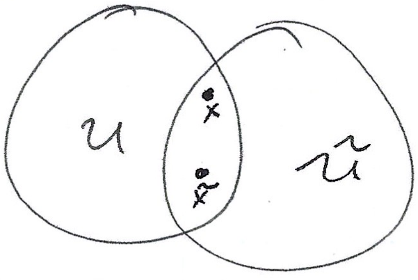
4.) (a.) show that if $W_e(\Sigma) = 1$ for all closed loops Σ then the gauge bundle is trivial?

[Hint: use Wilson lines to construct a global section].

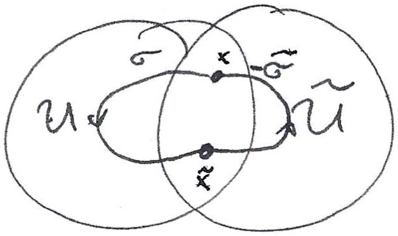
Let us follow the hint. Take two charts $\mathcal{U}, \tilde{\mathcal{U}}$ with $x, \tilde{x}$ in their intersection:



Let $\sigma, \tilde{\sigma}$ be paths from $x, \tilde{x}$ with

$$\sigma \subseteq \mathcal{U}$$
$$\tilde{\sigma} \subseteq \tilde{\mathcal{U}}$$

And define $\Sigma := \sigma - \tilde{\sigma}$ as a closed path from x to itself:



Let us now define Wilson lines.

$$W_e(\tilde{\sigma}, x, \tilde{x}) = \exp\{ie\int_\sigma A\}$$

$$\tilde{W}_e(\tilde{\sigma}, x, \tilde{x}) = \exp\{ie\int_{\tilde{\sigma}} \tilde{A}\}.$$

The name path Σ is closed by assumption so we

~~$1 = W_e(\Sigma)$~~

~~$= (W_e(\sigma, x, \tilde{x}) W_e(-\tilde{\sigma}, x$~~

~~$= W_e(\sigma - \tilde{\sigma}, x, \hat{x})$~~

~~$= \exp\{ie \int_{\sigma-\tilde{\sigma}}$~~

First, note that since $x, \tilde{x}$ is in the intersection, we have that

~~$\tilde{W}_e(\tilde{\sigma}, x, \tilde{x}) = g_{u\tilde{u}}(x) \exp\{ie \int_{\tilde{\sigma}} \tilde{A}\} g_{\tilde{u}u}(\tilde{x})$~~

I think my use of tildes messed me up.

We actually have

$$W_e(\tilde{\sigma}, x, \tilde{x}) = g_{u\tilde{u}}(x) \exp\underbrace{\{ie \int_{\tilde{\sigma}} \tilde{A}\} g_{\tilde{u}u}(\tilde{x})}$$

$$= \tilde{W}_e(\tilde{\sigma}, x, \tilde{x})$$

$$= g_{u\tilde{u}}(x) \tilde{W}_e(\tilde{\sigma}, x, \tilde{x})$$

Now, we can expand

$$1 = W_e(\Sigma)$$

$$= W_e(\sigma - \tilde{\sigma})$$

$$= W_e(\sigma, x, \tilde{x}) \tilde{W}_e(-\tilde{\sigma}, \tilde{x}, x)$$

But note that

$$\tilde{W}_e(-\tilde{\sigma}, \tilde{x}, x) = [\tilde{W}_e(\tilde{\sigma}, x, \tilde{x})]^{-1}$$

So that

$$1 = W_e(\sigma, x, \tilde{x}) [\tilde{W}_e(\tilde{\sigma}, x, \tilde{x})]^{-1}$$

$$\Rightarrow \boxed{W_e(\sigma, x, \hat{x}) = \tilde{W}_e(\tilde{\sigma}, x, \tilde{x})}$$

so the Wilson lines are path independent.

We further get that

$$W_e(\sigma, x, \tilde{x}) = \tilde{W}_e(\tilde{\sigma}, x, \tilde{x})$$

$$= g_{u\tilde{u}}(x)\, \tilde{W}(\tilde{\sigma}, x, \tilde{x})$$

Thus, if we define

$$g_u(x) = W_e(\sigma, x, \tilde{x})$$
$$g_{\tilde{u}}(x) = \tilde{W}_e(\tilde{\sigma}, x, \tilde{x})$$

We have two well-defined and continuous functions that are glued together by

$$\boxed{g_u(x) = g_{u\tilde{u}}(x)\, g_{\tilde{u}}(x)}$$

This gives a global section, and as such, the gauge bundle is trivial. □

(b.) Suppose that the characteristic class is trivial.

Show that the connection can be shifted to set $W_e(\Sigma) = 1$ for all Σ and therefore the gauge bundle is trivial

---

From class, we have

$$\Phi(S) = \frac{1}{e}\left[\int_S F - \delta\omega(S)\right]$$

$$= \frac{1}{e}\left[\int_S F - \omega(\partial S)\right]$$

where

$$\exp[ie\,\omega(\partial S)] = W_e(\partial S).$$

Suppose that ~~[illegible]~~ ϕ is trivial, i.e.

$$\phi \cong 0 \in H^2(M;\mathbb{Z}).$$

Then, ϕ must be an exact 2-cochain such that there is a 1-cochain Δ with

$$\phi = \delta\Delta$$

Since ϕ is integral, so too must Δ, that is,

$$\Delta(S) \in \mathbb{Z}.$$

From class, we know that w can be shifted by

$$w(\Sigma) \to w(\Sigma) + \tilde{e}\Delta'(\Sigma) + \delta\lambda(\Sigma)$$
$$= w(\Sigma) + \tilde{e}\Delta'(\Sigma) + \lambda(\partial\Sigma)$$

with $\Delta'(\Sigma) \in \mathbb{Z}$.

For $\partial\Sigma = 0$, this gives

$$w(\Sigma) \to w(\Sigma) + \tilde{e}\Delta'(\Sigma)$$

Which gives

$$W_e(\Sigma) = \exp\{iew(\Sigma)\} \to \exp\{ie(w(\Sigma)+\tilde{e}\Delta'(\Sigma)\}$$

But $e\tilde{e} = 2\pi$ so this is simply

$$\exp\{ie(w(\Sigma)+\tilde{e}\Delta'(\Sigma))\} = \exp\{iew(\Sigma) + 2\pi i\Delta'(\Sigma)\}$$
$$= W_e(\Sigma)\exp\{2\pi i\Delta'(\Sigma)\}$$

With $\Delta'(\Sigma) \in \mathbb{Z}$ we thus have

$$W_e(\Sigma) \to W_e(\Sigma)\exp\{2\pi i n\}$$
$$= W_e(\Sigma).$$

With this transformation and $\Sigma = \partial S$, we get

$$\phi(S) = \delta\Delta(S) \longrightarrow \delta\Delta(S) - \delta\Delta'(S)$$

$$= \Delta(\partial S) - \Delta'(\partial S)$$

If $\phi(S) = 0$ we find that

$$\boxed{\underline{\Delta(\partial S) - \Delta'(\partial S) = 0.}}$$

So that

$$w \to w + \tilde{e}\Delta + \delta\lambda.$$

Now if we have $\phi \equiv 0$ and S such that $\partial S = 0$, then

$$\phi(S) = \frac{1}{e}\left[\int_S F - \cancelto{0}{\omega(\partial S)}\right]$$

$$= \frac{1}{e}\left[\int_S F\right]$$

$$= \frac{1}{e}\oint_S F$$

$$= 0.$$

$$\Rightarrow \quad 0 = \oint_S F$$

So $F = d\omega$ for some 1-form ~~ω~~ ω

Any shift of the connection will be of the form

$$A_i \longrightarrow A_i + \delta A$$

Which implies that

$$F = dA_i \longrightarrow d(A_i + \delta A) = dA_i + d\delta A$$

$$= F + d\delta A$$

$$\cancel{= F + d\omega}$$

$$= d\omega + d\delta A$$

So if we take $\delta A = -\omega$, then

$$F \to d\omega + d\delta A = d\omega - d\omega = 0.$$

Thus, $F \to 0$ with $\delta A = -\omega$.

In general,

$$\phi(S) = \frac{1}{\tilde{e}}\left[\int_S F - w(\partial S)\right]$$

for $k\Sigma = \partial S$, $k \in \mathbb{Z}_{\geq 1}$ we have that

$$w(\Sigma) = \frac{1}{k} w(k\Sigma) = \frac{1}{k} w(\partial S) = \frac{1}{k}\int_S F - \frac{\tilde{e}}{k}\phi(S)$$

So our Wilson lines are given by

$$W_e(\Sigma) = \exp\{ie\,w(\Sigma)\} = \exp\left\{ie/k\int_S F - \frac{ie\tilde{e}}{k}\phi(S)\right\}$$

But, we know that since $\phi \cong 0$ we have

$$F = 0, \quad \phi(S) = 0$$

Thus,

$$W_e(\Sigma) = \exp\left\{\cancelto{0}{ie/k\,(0)} - \cancelto{0}{\frac{ie\tilde{e}}{k}(0)}\right\} = \exp\{0\} = I$$

Thus,

$$W_e(\Sigma) = 1$$

for all closed Σ.

Which from part (a.) means that the gauge bundle is trivial. □

(c.) Show that the characteristic class of a trivial gauge bundle is trivial:

$$\phi \cong 0 \in H^2(M;\mathbb{Z})$$

Therefore, the characteristic class is trivial if and only if the gauge bundle is trivial.

If the gauge bundle is trivial, we can set all transition functions to 1:

$$g_{ij}(x) = 1 \quad \forall i,j$$

Implying that

$$A_i = A_j + \cancel{g_{ij}\, dg_{ij}^{\dagger}} \rightarrow 0$$
$$= A_j$$

for all i,j.

Thus, we can globally define an A:

$$A = A_i \quad \forall i.$$

Thus, Wilson loops will have the form of

$$W_e(\partial S) = \exp\{ie\oint_{\partial S} A\}$$
$$= \exp\{ie\, w(\partial S)\}$$
$$\Rightarrow \boxed{w(\partial S) = \oint_{\partial S} A}$$

We then have that

$$\phi(S) = \frac{1}{e}\left[\int_S F - w(\partial S)\right]$$
$$= \frac{1}{e}\left[\int_S dA - w(\partial S)\right]$$
$$= \frac{1}{e}\left[\int_S dA - \oint_{\partial S} A\right]$$

$$= \frac{1}{e}\left[\int_S dA - \int_S dA\right]$$

$$= \frac{1}{e}[0]$$

$$= 0.$$

Thus, since S is any surface,
$\phi(S) = 0 \ \forall S \Rightarrow \phi \cong 0 \in H^2(M;\mathbb{Z})$

So if the gauge bundle is trivial, the characteristic class is as well.

When combined with parts (a) and (b), we see that

gauge bundle is trivial $\Longleftrightarrow \phi \cong 0 \in H^2(M;\mathbb{Z})$.

□